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ON THE EXISTENCE OF SOLUTIONS FOR FRACTIONAL DIFFERENTIAL INCLUSIONS WITH BOUNDARY CONDITIONS

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Abstract

We prove a Filippov type existence theorem for fractional differential inclusions defined by Caputo fractional derivative with boundary conditions by applying the contraction principle in the space of selections of the multifunction instead of the space of solutions. This approach allows to obtain also the Lipschitz dependence on the boundary conditions of the solution set of the problem considered.

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Key Words and Phrases: fractional differential inclusion, Caputo fractional derivative, boundary value problem, fixed point, contractive set-valued map

1. Introduction

In this note we study the following problem

$$D_c^\alpha x(t) \in F(t, x(t)) \quad a.e.\ ([0,1]), \tag{1.1}$$

$$x(0) = a_0, \quad x(1) = a_1, \tag{1.2}$$

where $\alpha \in (1,2)$, D_c^α is the Caputo fractional derivative, $F : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map and $a_0, a_1 \in \mathbf{R}$, $a_0, a_1 \neq 0$.

Differential equations with fractional order have recently proved to be strong tools in the modeling of many physical phenomena. As a consequence there was an intensive development of the theory of differential equations of fractional order ([2, 14, 16, 18] etc.). The study of fractional differential inclusions was initiated by El-Sayed and Ibrahim ([10]). Very recently several qualitative results for fractional differential inclusions were obtained in [1, 3, 12, 17] etc.. For a consistent bibliography on this topic, historical remarks and examples we refer to [1].

The present note is motivated by a recent paper of Chang and Nieto ([8]) where it is obtained an existence result for problem (1.1)-(1.2) using fixed point techniques when $F(.,.)$ has convex values. We note also that in [17] it is studied a fractional differential inclusion defined by Riemann-Liouville fractional derivative with Dirichlet boundary conditions. In contrast with the Riemann-Liouville fractional derivative, Caputo's fractional derivative allows the utilization of physically interpretable boundary/initial conditions.

The aim of our paper is to provide an additional existence result for problem (1.1)-(1.2) in the case when the values of $F(.,.)$ are not necessarily convex. More exactly, we prove a Filippov type result concerning the existence of solutions to the boundary value problem (1.1)-(1.2). We recall that for a differential inclusion defined by a Lipschitzian set-valued map with nonconvex values, Filippov's theorem ([11]) consists in proving the existence of a solution starting from a given "quasi" or "almost" solution. Moreover, the result provides an estimation between the "quasi" solution and the solution obtained.

Our approach is different from the usual ones (e.g., [1]) and consists in the application of the set-valued contraction principle in the space of selections of the set-valued map instead of the space of solutions. At the same time, our approach allows to prove that the map that associates to a given boundary condition $(a_0, a_1) \in \mathbf{R}^2$ the set of solutions of problem (1.1) starting from (a_0, a_1) depends Lipschitz-continuously on the boundary condition.

We note that the idea of applying the set-valued contraction principle due to Covitz and Nadler ([9]) in the space of derivatives of the solutions belongs to Tallos ([13], [19]) and it was already used for similar results obtained for other classes of differential inclusions ([5-7]).

The paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in the sequel and in Section 3 we prove the main results.

2. Preliminaries

In this short section we sum up some basic facts that we are going to use later.

Let (X,d) be a metric space and consider a set valued map T on X with nonempty closed values in X. T is said to be a λ -contraction if there exists $0<\lambda<1$ such that:

$$d_H(T(x),T(y)) \leq \lambda d(x,y) \quad \forall x,y \in X,$$

where $d_H(.,.)$ denotes the Pompeiu-Hausdorff distance. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A,B \subset X$ is defined by

$$d_H(A,B)=\max\{d^*(A,B),d^*(B,A)\}, \quad d^*(A,B)=\sup\{d(a,B);a\in A\},$$

where $d(x,B)=\inf_{y\in B} d(x,y)$.

If X is complete, then every set valued contraction has a fixed point, i.e. a point $z\in X$ such that $z\in T(z)$ ([9]).

We denote by $Fix(T)$ the set of all fixed points of the set-valued map T. Obviously, $Fix(T)$ is closed.

Proposition 2.1. *([15]) Let X be a complete metric space and suppose that T_1,T_2 are λ-contractions with closed values in X. Then*

$$d_H(Fix(T_1),Fix(T_2)) \leq \frac{1}{1-\lambda}\sup_{z\in X} d(T_1(z),T_2(z)).$$

Let $I:=[0,1]$, denote by $C(I,\mathbf{R})$ the Banach space of all continuous functions from I to $\mathbf{R}$ endowed with the norm $||x||_C=\sup_{t\in I}|x(t)|$ and by $L^1(I,\mathbf{R})$ we denote the Banach space of Lebesgue integrable functions $u(.):I\to\mathbf{R}$ endowed with the norm $||u||_1=\int_0^1|u(t)|dt$.

Definition 2.2. (see [14]) a) *The fractional integral of order $\alpha>0$* of a Lebesgue integrable function $f(.):(0,\infty)\to\mathbf{R}$ is defined by

$$I^\alpha f(t)=\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds,$$

provided the right-hand side is pointwise defined on $(0,\infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}dt$.

b) *The Caputo fractional derivative of order* $\alpha > 0$ of a function $f(.) : [0, \infty) \to \mathbf{R}$ is defined by

$$D_c^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{-\alpha+n-1} f^{(n)}(s) ds,$$

where $n = [\alpha] + 1$. It is assumed implicitly that $f(.)$ is n times differentiable whose n-th derivative is absolutely continuous.

We recall (e.g., [14]) that if $\alpha > 0$ and $f(.) \in C(I, \mathbf{R})$ or $f(.) \in L^\infty(I, \mathbf{R})$ then $(D_c^\alpha I^\alpha f)(t) \equiv f(t)$.

DEFINITION 2.3. (see [8]) A function $x(.) \in C(I, \mathbf{R})$ is called a solution of problem (1.1)-(1.2) if there exists a function $v(.) \in L^1(I, \mathbf{R})$ such that $v(t) \in F(t, x(t))$, *a.e.* (I) and

$$x(t) = a_0 + (a_1 - a_0)t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds - \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s) ds. \tag{2.1}$$

REMARK 2.4. If we denote $P_a(t) = a_0 + (a_1 - a_0)t$, $a = (a_0, a_1) \in \mathbf{R}^2$ and

$$G(t,s) := \frac{1}{\Gamma(\alpha)} \begin{cases} -t(1-s)^{\alpha-1} + (t-s)^{\alpha-1}, & \text{if} \quad 0 \le s < t \le 1, \\ -t(1-s)^{\alpha-1}, & \text{if} \quad 0 \le t < s \le 1, \end{cases}$$

then the solution in (2.1) of problem (1.1)-(1.2) can be rewritten as

$$x(t) = P_a(t) + \int_0^1 G(t,s) f(s) ds. \tag{2.2}$$

Note that $|G(t,s)| \le \frac{2}{\Gamma(\alpha)}$ $\forall t, s \in I$ and that if $a = (a_0, a_1), b = (b_0, b_1) \in \mathbf{R}^2$ and we set $||a|| = |a_0| + |a_1|$ then

$$|P_a(t) - P_b(t)| \le ||a - b||.$$

In the sequel we assume the following conditions on F.

HYPOTHESIS 2.5. i) $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *has nonempty closed values and for every* $x \in \mathbf{R}$ $F(., x)$ *is measurable.*

ii) *There exists* $L(.) \in L^1(I, \mathbf{R})$ *such that for almost all* $t \in I, F(t, \cdot)$ *is L(t)-Lipschitz in the sense that*

$$d_H(F(t,x), F(t,y)) \le L(t)|x - y| \quad \forall \ x, y \in \mathbf{R}$$

and $d(0, F(t, 0)) \le L(t)$ *a.e.* (I).

3. The main results

We are able now to prove our main result.

THEOREM 3.1. *Assume that Hypothesis 2.5 is satisfied and* $\frac{2}{\Gamma(\alpha)}||L||_1 < 1$. *Let* $y(.) \in C(I,\mathbf{R})$ *be such that there exists* $q(.) \in L^1(I,\mathbf{R})$ *with* $d(D_c^\alpha y(t), F(t,y(t))) \leq q(t)$, *a.e.* (I). *Denote* $b_0 = y(0)$, $b_1 = y(1)$. *Then for every* $\varepsilon > 0$ *there exists* $x(.)$ *a solution of (1.1)-(1.2) satisfying for all* $t \in I$

$$|x(t)-y(t)| \leq \frac{\Gamma(\alpha)}{\Gamma(\alpha)-2||L||_1}||a-b|| + \frac{2}{\Gamma(\alpha)-2||L||_1}\int_0^1 q(t)dt + \varepsilon. \quad (3.1)$$

P r o o f. For $u(.) \in L^1(I,\mathbf{R})$ define the following set valued maps:

$$M_u^a(t) = F(t, P_a(t) + \int_0^1 G(t,s)u(s)ds), \quad t \in I, \quad (3.2)$$

$$T_a(u) = \{\phi(.) \in L^1(I,\mathbf{R}); \quad \phi(t) \in M_u^a(t) \quad a.e.\ (I)\}. \quad (3.3)$$

It follows from the definition and (2.2) that $x(.)$ is a solution of (1.1)-(1.2) if and only if $D_c^\alpha x(.)$ is a fixed point of $T_a(.)$.

We shall prove first that $T_a(u)$ is nonempty and closed for every $u \in L^1(I,\mathbf{R})$. The fact that the set valued map $M_u^a(.)$ is measurable is well known. For example the map $t \to P_a(t) + \int_0^1 G(t,s)u(s)ds$ can be approximated by step functions and we can apply Theorem III. 40 in [4]. Since the values of F are closed with the measurable selection theorem (Theorem III.6 in [4]) we infer that $M_u^a(.)$ admits a measurable selection ϕ. One has

$$|\phi(t)| \leq d(0,F(t,0)) + d_H(F(t,0), F(t,P_a(t) + \int_0^1 G(t,s)u(s)ds)$$

$$\leq L(t)(1 + |P_a(t)| + \frac{2}{\Gamma(\alpha)}\int_0^1 |u(s)|ds),$$

which shows that $\phi \in L^1(I,\mathbf{R})$ and $T_a(u)$ is nonempty.

On the other hand, the set $T_a(u)$ is also closed. Indeed, if $\phi_n \in T_a(u)$ and $||\phi_n - \phi||_1 \to 0$ then we can pass to a subsequence ϕ_{n_k} such that $\phi_{n_k}(t) \to \phi(t)$ for a.e. $t \in I$, and we find that $\phi \in T(u)$.

We show next that $T_a(.)$ is a contraction on $L^1(I,\mathbf{R})$.

Let $u, v \in L^1(I,\mathbf{R})$ be given, $\phi \in T_a(u)$ and let $\delta > 0$. Consider the following set-valued map:

$$H(t) = M_v^a(t) \cap \{x \in \mathbf{R};\ |\phi(t) - x| \leq L(t)|\int_0^1 G(t,s)(u(s)-v(s))ds| + \delta\}.$$

From Proposition III.4 in [4], $H(.)$ is measurable and from Hypothesis 2.5 ii) $H(.)$ has nonempty closed values. Therefore, there exists $\psi(.)$ a measurable selection of $H(.)$. It follows that $\psi \in T_a(v)$ and according with the definition of the norm we have

$$||\phi - \psi||_1 = \int_0^1 |\phi(t) - \psi(t)|dt \leq \int_0^1 L(t)(\int_0^1 |G(t,s)|.|u(s) - v(s)|ds)dt$$

$$+\int_0^1 \delta dt = \int_0^1 (\int_0^1 L(t)|G(t,s)|dt)|u(s)-v(s)|ds+\delta \leq \frac{2}{\Gamma(\alpha)}||L||_1||u-v||_1+\delta.$$

Since $\delta > 0$ was chosen arbitrary, we deduce that

$$d(\phi, T_a(v)) \leq \frac{2}{\Gamma(\alpha)}||L||_1||u - v||_1.$$

Replacing u by v we obtain

$$d_H(T_a(u), T_a(v)) \leq \frac{2}{\Gamma(\alpha)}||L||_1||u - v||_1,$$

thus $T(.)$ is a contraction on $L^1(I, \mathbf{R})$.

We consider next the following set-valued maps

$$F_1(t,x) = F(t,x) + q(t)[-1,1], \quad (t,x) \in I \times \mathbf{R},$$

$$M^b_{1,u}(t) = F_1(t, P_b(t) + \int_0^1 G(t,s)u(s)ds), \quad t \in I, \quad u(.) \in L^1(I, \mathbf{R}),$$

$$T^1_b(u) = \{\psi(.) \in L^1(I, \mathbf{R}); \quad \psi(t) \in M^b_{1,u}(t) \quad a.e.\ (I)\}.$$

Obviously, $F_1(.,.)$ satisfies Hypothesis 2.5.

Repeating the previous step of the proof we obtain that T^1_b is also a $\frac{2}{\Gamma(\alpha)}||L||_1$-contraction on $L^1(I, \mathbf{R})$ with closed nonempty values.

We prove next the following estimate

$$d_H(T_a(u), T^1_b(u)) \leq ||a - b||.||L||_1 + \int_0^1 q(t)dt. \tag{3.4}$$

Let $\phi \in T_a(u), \delta > 0$ and define

$$H_1(t) = M^b_{1,u}(t) \cap \{z \in \mathbf{R}; \quad |\phi(t) - z| \leq L(t)|P_a(t) - P_b(t)| + q(t) + \delta\}.$$

With the same arguments used for the set valued map $H(.)$, we deduce that $H_1(.)$ is measurable with nonempty closed values. Hence let $\psi(.)$ be a measurable selection of $H_1(.)$. It follows that $\psi \in T_b^1(u)$ and one has

$$||\phi - \psi||_1 = \int_0^1 |\phi(t) - \psi(t)|dt \leq \int_0^1 [L(t)|P_a(t) - P_b(t)| + q(t) + \delta]dt$$

$$\leq ||a-b|| \int_0^1 L(t) + \int_0^1 q(t) + \delta.$$

Since δ is arbitrary, as above we obtain (3.4).

We apply Proposition 2.1 and we infer that

$$d_H(Fix(T_a), Fix(T_b^1)) \leq \frac{1}{1 - \frac{2}{\Gamma(\alpha)}||L||_1}[||a-b||.||L||_1 + \int_0^1 q(t)dt].$$

Since $D_c^\alpha y(.) \in Fix(T_b^1)$ it follows that there exists $u(.) \in Fix(T_a)$ such that for any $\varepsilon > 0$

$$||D_c^\alpha y - u||_1 \leq \frac{1}{1 - \frac{2}{\Gamma(\alpha)}||L||_1}[||a-b||.||L||_1 + \int_0^1 q(t)dt] + \frac{\Gamma(\alpha)\varepsilon}{2}.$$

We define $x(t) = P_a(t) + \int_0^1 G(t,s)u(s)ds$, $t \in I$ and we have

$$|x(t) - y(t)| \leq |P_a(t) - P_b(t)| + \int_0^1 |G(t,s)|.|u(s) - D_c^\alpha y(s)|ds$$

$$\leq ||a-b|| + \frac{2}{\Gamma(\alpha)}||u - D_c^\alpha y||_1 \leq \frac{\Gamma(\alpha)}{\Gamma(\alpha) - 2||L||_1}||a-b|| + \frac{2}{\Gamma(\alpha) - 2||L||_1}||q||_1 + \varepsilon,$$

which completes the proof. ■

Remark 3.2. The assumption in Theorem 3.1 is satisfied, in particular, for $y(.) = 0$ and therefore, via Hypothesis 2.5, with $q(.) = L(.)$. In this case, Theorem 3.1 provides an existence result for problem (1.1)-(1.2) together with a priori bounds for the solution. More precisely, the estimate (3.1) is in this case

$$|x(t)| \leq \frac{\Gamma(\alpha)||a|| + 2||L||_1}{\Gamma(\alpha) - 2||L||_1} + \varepsilon, \quad \forall t \in I.$$

On the other hand, in our approach we obtain a "pointwise" estimate from a norm estimate. Therefore an estimate as in (3.1) is not comparable with estimates obtained using the classical Filippov construction (e.g., [17]).

Let us denote by $\mathcal{S}(a)$ the set of all solutions of problem (1.1)-(1.2), where as above $a = (a_0, a_1) \in \mathbf{R}^2$. As we already pointed out the idea in the proof of Theorem 3.1 can be adapted in order to prove the Lipschitz continuity of the solution map $\mathcal{S}(.)$.

PROPOSITION 3.3. *Let Hypothesis 2.5 be satisfied and let* $l := \frac{2}{\Gamma(\alpha)}||L||_1 < 1$. *Then the map* $a \to \mathcal{S}(a)$ *is Lipschitz-continuous on* $\mathbf{R}^2$ *with nonempty closed values in* $C(I, \mathbf{R})$.

P r o o f. Let $a, b \in \mathbf{R}^2$. For $u(.) \in L^1(I, \mathbf{R})$, let $M_u^a(.), T_a(.)$ be defined as in (3.2) and (3.3), respectively. As in the proof of Theorem 3.1, $T_a(.)$ is a l-contraction on $L^1(I, \mathbf{R})$ with nonempty and closed values.

Consequently, $T_a(.)$ admits a fixed point $u(.) \in L^1(I, \mathbf{R})$. By setting $x(t) = P_a(t) + \int_0^1 G(t,s)u(s)ds$ we have that $x(.) \in \mathcal{S}(a)$.

Since $Fix(T_a)$ is closed we also get that $\mathcal{S}(a) \subset C(I, \mathbf{R})$ is a closed set. Moreover, as in the proof of Theorem 3.1,

$$d_H(T_a(u), T_b(u)) \leq ||a - b||.||L||_1 \tag{3.5}$$

for any $a, b \in \mathbf{R}^2$ and $u(.) \in L^1(I, \mathbf{R})$.

Using Proposition 2.1, from (3.3) it follows that

$$d_H(Fix(T_a), Fix(T_b)) \leq \frac{||L||_1}{1-l}||a - b||.$$

Since $u(.) \in Fix(T_a)$, for any $\varepsilon > 0$ there exists $v(.) \in Fix(T_b)$ such that

$$||u(.) - v(.)||_1 \leq \frac{||L||_1}{1-l}||a - b|| + \varepsilon.$$

Put $y(t) = P_b(t) + \int_0^1 G(t,s)u(s)ds$. Then $y(.) \in \mathcal{S}(b)$ and

$$||x - y||_C \leq ||a - b|| + \sup_{t,s \in I} |G(t,s)|||u - v||_1 \leq \frac{1}{1-l}||a - b|| + \frac{2\varepsilon}{\Gamma(\alpha)}.$$

Hence

$$d(x, \mathcal{S}(b)) \leq \frac{1}{1-l}||a - b|| + \frac{2\varepsilon}{\Gamma(\alpha)}, \quad \forall \varepsilon > 0,$$

which means $d(x, \mathcal{S}(b)) \leq \frac{1}{1-l}||a - b||$. By interchanging a and b we finally obtain $d_H(\mathcal{S}(a), \mathcal{S}(b)) \leq \frac{1}{1-l}||a - b||$, that completes the proof. ■

REMARK 3.4. Similar results as in Theorem 3.1 and Proposition 3.3 may be obtained for initial value problems associated to fractional differential inclusions of the form

$$D_c^\alpha x(t) \in F(t, x(t)), \quad \alpha \in (1,2),$$

$$x(0) = a_0, \quad x'(0) = a_1.$$

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